Pre-Leaving Certificate Examination, 2015

Applied Mathematics

Higher Level
Marking Scheme (300 marks)

Six questions to be answered. All questions carry equal marks. (6 × 50m)

1. (a) A train travels a distance d from rest at one station to rest at another station. The train travels for the first part of its journey with a constant acceleration f_1. It then immediately decelerates to rest at the second station with a constant deceleration f_2.

Show that the total time taken is $\sqrt{2d\left(\frac{1}{f_1}+\frac{1}{f_2}\right)}$.

(25)

Let v be the maximum speed attained.

Then

1: $f_1 = \frac{v}{t_1}$

$t_1 = \frac{v}{f_1}$ ... (5m)

2: $f_2 = \frac{v}{t_2}$

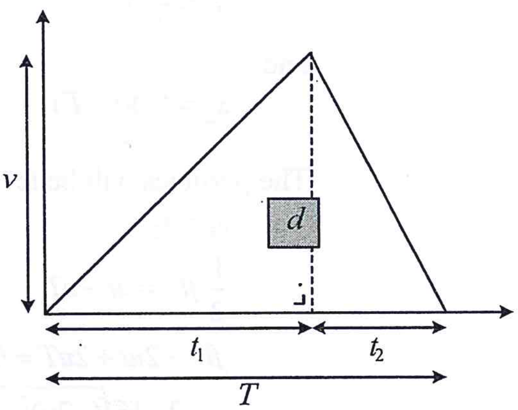
$t_2 = \frac{v}{f_2}$ ... (5m)

3: $T = t_1 + t_2$

$T = \frac{v}{f_1} + \frac{v}{f_2}$

$T = v\left(\frac{1}{f_1}+\frac{1}{f_2}\right)$ ... (5m)

$v = \frac{T}{\left(\frac{1}{f_1}+\frac{1}{f_2}\right)}$



Further answers overleaf

4: $d = \frac{1}{2}Tv$ ... (5m)

$$2d = T.\frac{T}{\left(\frac{1}{f_1}+\frac{1}{f_2}\right)}$$

$$2d\left(\frac{1}{f_1}+\frac{1}{f_2}\right) = T^2$$

$$T = \sqrt{2d\left(\frac{1}{f_1}+\frac{1}{f_2}\right)} \quad \text{... (5m)}$$

(b) A particle, P, starts from rest at a point A and moves with constant acceleration f in a straight line. A time T, after P starts from A, a second particle, Q, starts from A and moves in the same direction along the same straight line as P.

Q moves with a constant speed of u.

(i) Prove that Q will overtake P if $u > 2fT$. **(20)**

Let $t > T$. Let s_1 and s_2 be the distances travelled by P and Q in time t.

Then

$$s_1 = (0)(t) + \frac{1}{2}(f)(t)^2$$

$$s_1 = \frac{1}{2}ft^2$$

and

$$s_2 = (u)(t-T) \quad \text{... (5m)}$$

The particles will be level when

$$s_1 = s_2 \quad \text{... (5m)}$$

$$\frac{1}{2}ft^2 = ut - uT$$

$$ft^2 - 2ut + 2uT = 0$$

$$t = \frac{2u \pm \sqrt{(-2u)^2 - 4(f)(2uT)}}{2f}$$

$$t = \frac{2u \pm 2\sqrt{u^2 - 2fuT}}{2f}$$

$$t = \frac{u \pm \sqrt{u^2 - 2fuT}}{f} \quad \text{... (5m)}$$

For Q to overtake P, this equation must have real solutions, i.e. $b^2 - 4ac > 0$

$$u^2 - 2fuT > 0$$

$$u^2 > 2fuT$$

$$u > 2fT$$ ... (5m)

(ii) Assuming Q does overtake P, i.e. that $u > 2fT$, express in terms of u, f and T the length of time for which Q is ahead of P. **(5)**

Let $t_1 = \dfrac{u - \sqrt{u^2 - 2fuT}}{f}$ and $t_2 = \dfrac{u + \sqrt{u^2 - 2fuT}}{f}$.

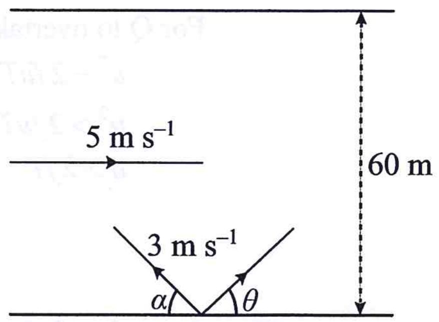
The length of time Q is ahead of P is

$$t_2 - t_1 = \left(\frac{u + \sqrt{u^2 - 2fuT}}{f}\right) - \left(\frac{u - \sqrt{u^2 - 2fuT}}{f}\right)$$

$$= \frac{2\sqrt{u^2 - 2fuT}}{f}$$ ... (5m)

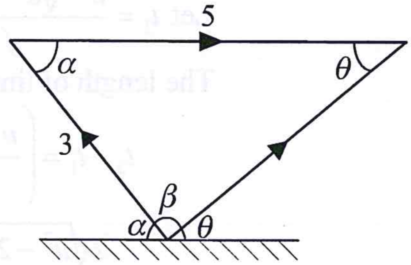
2. **(a)** A man can swim at 3 m s $^{-1}$ in still water. He swims across a river of width 60 metres. The river flows with a constant speed of 5 m s $^{-1}$ parallel to the straight banks.

He swims at an angle α to the upstream direction but ends up going at an angle θ to the downstream direction.



(i) Show that $\tan\theta = \dfrac{3\sin\alpha}{5-3\cos\alpha}$. **(10)**

Let $\vec{i}$ and $\vec{j}$ be unit vectors in the direction of the river flow and perpendicular to the banks respectively.



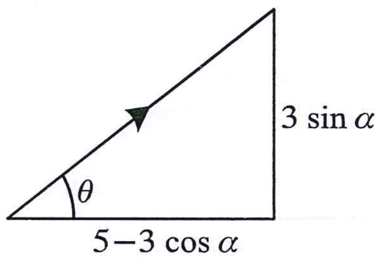
Then

$$\vec{v}_R = 5\vec{i}$$

$$\vec{v}_{MR} = -3\cos\alpha\vec{i} + 3\sin\alpha\vec{j}$$

and

$$\vec{v}_M = \vec{v}_{MR} + \vec{v}_R$$
$$= (-3\cos\alpha\vec{i} + 3\sin\alpha\vec{j}) + (5\vec{i})$$
$$= (5-3\cos\alpha)\vec{i} + 3\sin\alpha\vec{j} \qquad \text{... (5m)}$$



If this makes an angle θ with the downstream direction, then

$$\tan\theta = \frac{3\sin\alpha}{5-3\cos\alpha} \qquad \text{... (5m)}$$

(ii) Find the time taken for the man to cross by the shortest path. **(15)**

Let β be the angle between $\vec{v}_{MR}$ and $\vec{v}_M$. Then by the Sine rule,

$$\frac{\sin\theta}{3} = \frac{\sin\beta}{5}$$

$$\sin\theta = \frac{3\sin\beta}{5}$$

For the shortest path, we require the largest possible value of θ.
This is when $\beta = 90°$
and

$$\sin\theta = \frac{3}{5} \quad \text{... (5m)}$$

then

$$|\vec{v}_M| = \sqrt{5^2 - 3^2} = 4$$

and

$$\sin\alpha = \frac{4}{5} \quad \text{... (5m)}$$

Then the speed directly across the river is

$$3\sin\alpha = 3\left(\frac{4}{5}\right) = 2\cdot 4$$

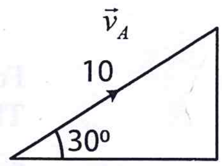
Thus the time to cross the river by the shortest path is

$$\frac{60}{2\cdot 4} = 25 \text{ s} \quad \text{... (5m)}$$

(b) Ship A is travelling with a constant speed of 10 m s $^{-1}$ in the direction 30° north of east. At midday, ship B is 10 km due east of ship A, and is travelling in a straight line with a constant speed of v m s $^{-1}$.

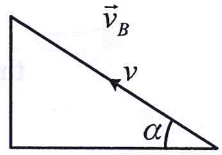
(i) Calculate the minimum possible value of v if B is to intercept A. **(20)**

$$\vec{v}_A = (10\cos 30°)\vec{i} + (10\sin 30°)\vec{j}$$
$$\vec{v}_A = 5\sqrt{3}\,\vec{i} + 5\vec{j} \qquad \text{... (5m)}$$



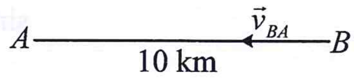
Let B travel in the direction α north of west.
Then

$$\vec{v}_B = -(v\cos\alpha)\vec{i} + (v\sin\alpha)\vec{j}$$



Thus

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$
$$= (-v\cos\alpha\,\vec{i} + v\sin\alpha\,\vec{j})$$
$$-\left(5\sqrt{3}\,\vec{i} + 5\vec{j}\right)$$
$$= (-v\cos\alpha - 5\sqrt{3})\vec{i}$$
$$+(v\sin\alpha - 5)\vec{j} \qquad \text{... (5m)}$$



For interception, $\vec{v}_{BA}$ must point due west, i.e.

$$v\sin\alpha - 5 = 0$$
$$\sin\alpha = \frac{5}{v} \qquad \text{... (5m)}$$

For this to be possible,

$$\sin\alpha \leq 1$$
$$\frac{5}{v} \leq 1$$
$$v \geq 5$$

Thus the minimum value of v for interception is 5. ... (5m)

(ii) If $v = 6$, show that B can travel in either of two directions to intercept A, and find these directions, correct to the nearest degree. **(5)**

If $v = 6$, then interception occurs when

$$6\sin\alpha - 5 = 0$$
$$\sin\alpha = \frac{5}{6}$$
$$\alpha = \sin^{-1}\frac{5}{6} = 56° \text{ or } \alpha = 180° - 56° = 124° \qquad \text{... (5m)}$$

3. **(a)** A particle is projected from a point on a horizontal plane with speed 21 m s $^{-1}$ at an angle α to the horizontal. The particle then strikes a small target whose horizontal and vertical distances from the point of projection are 30 m and 10 m respectively.

Find **(i)** the two possible values of $\tan\alpha$ **(15)**

Given: for the same value of t,

$s_x = 30$ and $s_y = 10$

$s_x = 30: \quad 21\cos\alpha . t = 30$

$t = \dfrac{10}{7\cos\alpha}$ ... (5m)

$s_y = 10: \quad 21\sin\alpha\left(\dfrac{10}{7\cos\alpha}\right) - \dfrac{g}{2}\left(\dfrac{100}{49\cos^2\alpha}\right) = 10$ ... (5m)

$30\tan\alpha - 10\sec^2\alpha = 10$

$3\tan\alpha - (1+\tan^2\alpha) = 1$

$\tan^2\alpha - 3\tan\alpha + 2 = 0$

$(\tan\alpha - 1)(\tan\alpha - 2) = 0$

$\tan\alpha = 1$ or $\tan\alpha = 2$ ... (5m)

(ii) the two possible times taken to strike the target. **(10)**

If $\tan\alpha = 1$, then $\alpha = 45°$ and $\cos\alpha = \dfrac{1}{\sqrt{2}}$. The time taken is.

$$\frac{10}{7\left(\frac{1}{\sqrt{2}}\right)} = \frac{10\sqrt{2}}{7} = 2 \cdot 02 \text{ s}$$ ... (5m)

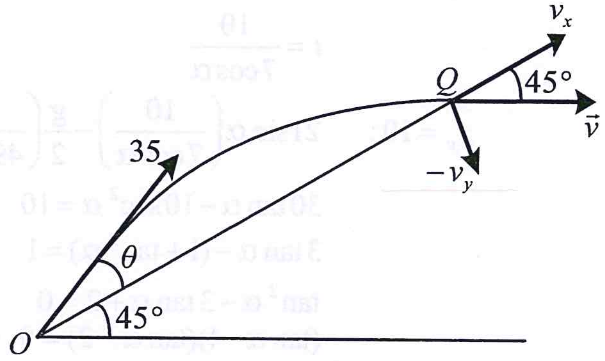
If $\tan\alpha = 2$, then $\cos\alpha = \dfrac{1}{\sqrt{5}}$. The time taken is

$$\frac{10}{7\left(\frac{1}{\sqrt{5}}\right)} = \frac{10\sqrt{5}}{7} = 3 \cdot 19 \text{ s}.$$ ... (5m)

(b) A particle is projected up an inclined plane from a point O, with initial speed of 35 m s^{-1}. The line of projection makes an angle θ with the inclined plane and the plane is inclined at an angle of 45° to the horizontal. The plane of projection is vertical and contains the line of greatest slope.

The particle is moving horizontally when it strikes the inclined plane at Q.

(i) Show that $\tan\theta = \dfrac{1}{3}$. **(15)**



$$v_x = 35\cos\theta - g\sin 45°.t \text{ and } v_y = 35\sin\theta - g\cos 45°.t$$

$$v_x = 35\cos\theta - \frac{gt}{\sqrt{2}} \text{ and } v_y = 35\sin\theta - \frac{gt}{\sqrt{2}}$$

Also, time of flight:

$$s_y = 0$$

$$35\sin\theta.t - \frac{gt^2}{2\sqrt{2}} = 0$$

$$t = \frac{70\sqrt{2}\sin\theta}{g}$$ ... (5m)

At the time of flight:

$$\tan 45° = \frac{-v_y}{v_x}$$

$$v_x = -v_y$$ ... (5m)

$$35\cos\theta - \frac{g}{\sqrt{2}}\left(\frac{70\sqrt{2}\sin\theta}{g}\right) = -\left(35\sin\theta - \frac{g}{\sqrt{2}}\left(\frac{70\sqrt{2}\sin\theta}{g}\right)\right)$$

$$35\cos\theta - 70\sin\theta = -35\sin\theta + 70\sin\theta$$

$$35\cos\theta = 105\sin\theta$$

$$\frac{35}{105} = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{3}$$ ... (5m)

(ii) Find $|OQ|$. **(10)**

$$\tan\theta = \frac{1}{3}$$

Then

$$\sin\theta = \frac{1}{\sqrt{10}} \text{ and } \cos\theta = \frac{3}{\sqrt{10}}$$

Then time of flight

$$= \frac{70\sqrt{2}}{g}\left(\frac{1}{\sqrt{10}}\right) = \frac{10\sqrt{5}}{7} \qquad \text{... (5m)}$$

and

$$|OQ| = s_x \text{ at time of flight}$$

$$|OQ| = 35\left(\frac{3}{\sqrt{10}}\right)\left(\frac{10\sqrt{5}}{7}\right) - \frac{g}{2\sqrt{2}}\left(\frac{10\sqrt{5}}{7}\right)^2 = 75\sqrt{2} - \frac{50}{\sqrt{2}}$$

$$|OQ| = 50\sqrt{2}$$

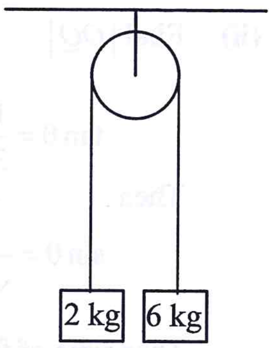
$$|OQ| = 70 \cdot 71 \text{ m} \qquad \text{... (5m)}$$

4. **(a)** Two particles of masses 2 kg and 6 kg are connected by a light inextensible string passing over a fixed smooth pulley.

Initially the two particles are at rest at the same horizontal level.

The system is released from rest.

The 6 kg particle takes 2 seconds to strike horizontal ground.



Find **(i)** the initial height of the particles above the ground

(15)

By Newton's 2nd Law:

$$2a = T - 2g$$

and

$$4a = 4g - T \qquad \text{... (5m)}$$

adding

$$6a = 2g$$

$$a = \frac{g}{3} \qquad \text{... (5m)}$$

Let h be the height of the particles above the ground.

Then

$$s = ut + \frac{1}{2}at^2$$

$$h = (0)(2) + \frac{1}{2}\left(\frac{g}{3}\right)(2)^2$$

$$h = \frac{2g}{3} \qquad \text{... (5m)}$$

(ii) the greatest height above the ground to which the 2 kg mass rises. **(10)**

After 2 seconds,

$$v = u + at$$

$$v = (0) + \left(\frac{g}{3}\right)(2) = \frac{2g}{3}$$

When the 4 kg mass strikes the ground, the 2 kg mass is at a height of

$$2h = \frac{4g}{3} \text{ above the ground.} \qquad \text{... (5m)}$$

Let h_1 be the further height the 2 kg mass rises to before coming to rest. Then

with $a = -g$ and $u = \frac{2g}{3}$

$$v^2 = u^2 + 2as$$

$$(0)^2 = \left(\frac{2g}{3}\right)^2 + 2(-g)h_1$$

$$2gh_1 = \frac{4g^2}{9}$$

$$h_1 = \frac{2g}{9}$$

The maximum height of the 2 kg mass is

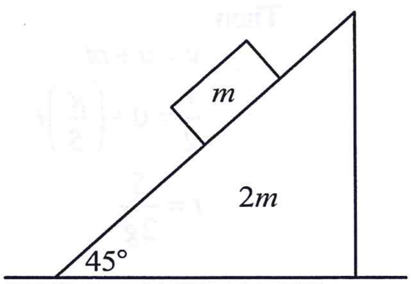
$$\frac{4g}{3} + \frac{2g}{9} = \frac{14g}{9}$$

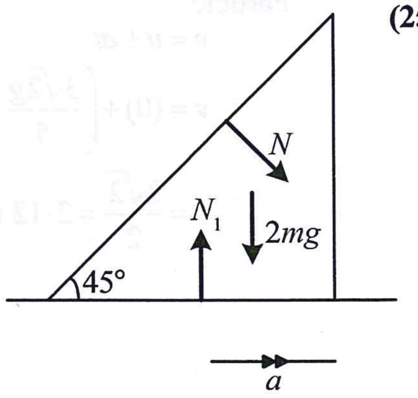
$$= 15 \cdot 24 \text{ m} \qquad \text{... (5m)}$$

(b) A smooth wedge, of mass $2m$ and slope 45°, rests on a smooth horizontal plane. A particle of mass m is placed on the inclined face of the wedge.

The system is released from rest.

Find the speed of the mass m relative to the wedge, when the speed of the wedge is $0 \cdot 5 \text{ m s}^{-1}$. **(25)**





Wedge

$$(2m)a = N \sin 45°$$

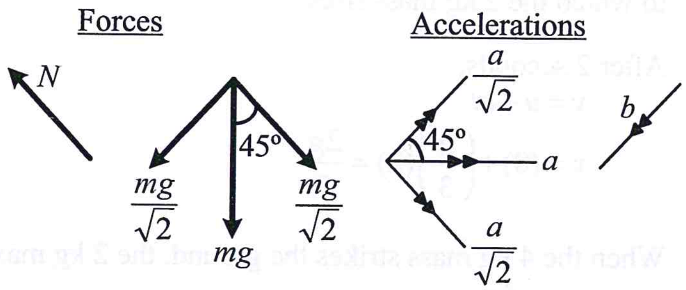
$$2ma = \frac{N}{\sqrt{2}}$$

$$ma = \frac{N}{2\sqrt{2}}$$

$$N = 2\sqrt{2}ma \qquad \text{... (5m)}$$

Further answers overleaf

Particle



⤡ $$m\left(\frac{a}{\sqrt{2}}\right)=\frac{mg}{\sqrt{2}}-N$$ ... (5m)

$$m\left(\frac{a}{\sqrt{2}}\right)=\frac{mg}{\sqrt{2}}-2\sqrt{2}ma$$

$$ma=mg-4ma$$

$$5a=g$$

$$a=\frac{g}{5}$$ ... (5m)

⤢ $$m\left(b-\frac{a}{\sqrt{2}}\right)=\frac{mg}{\sqrt{2}}$$

$$b-\frac{g}{5\sqrt{2}}=\frac{g}{\sqrt{2}}$$

$$b=\frac{3\sqrt{2}g}{5}$$ ... (5m)

Wedge

$$u=0,\ v=0\cdot5,\ a=\frac{g}{5}.$$

Then

$$v=u+at$$

$$\frac{1}{2}=0+\left(\frac{g}{5}\right)t$$

$$t=\frac{5}{2g}$$

Particle:

$$v=u+at$$

$$v=(0)+\left(\frac{3\sqrt{2}g}{5}\right)\left(\frac{5}{2g}\right)$$

$$v=\frac{3\sqrt{2}}{2}=2\cdot12\ \text{m s}^{-1}$$ ... (5m)

5. **(a)** A smooth sphere A, of mass m_1, collides directly with a smooth sphere B, of mass m_2 which is at rest on a smooth horizontal table. The coefficient of restitution between the spheres is e_1.

The line of centres of the spheres is at right angles to a smooth vertical cushion at the edge of the table. Sphere B then strikes the cushion and rebounds.

The coefficient of restitution between sphere B and the cushion is e_2.

Show that there will be no further impact between the spheres if $m_1(1+e_2+e_1e_2)<e_1m_2$. **(25)**

<u>Collision 1:</u> Two spheres

u	m	v
u	m_1	v_1
0	m_2	v_2

PCM: $m_1u+m_2(0)=m_1v_1+m_2v_2$ ... (5m)

$$m_1v_1+m_2v_2=m_1u$$

NEL: $v_1-v_2=-e_1u$ ... (5m)

Then

$$m_1v_1+m_2v_2=m_1u$$
$$m_2v_1-m_2v_2=-e_1m_2u$$
$$(m_1+m_2)v_1=(m_1-e_1m_2)u$$
$$v_1=\frac{m_1-e_1m_2}{m_1+m_2}u$$

also

$$m_1v_1+m_2v_2=m_1u$$
$$-m_1v_1+m_1v_2=e_1m_1u$$
$$(m_1+m_2)v_2=(m_1+e_1m_1)u$$
$$v_2=\frac{m_1(1+e_1)}{m_1+m_2}u \quad \text{... (5m)}$$

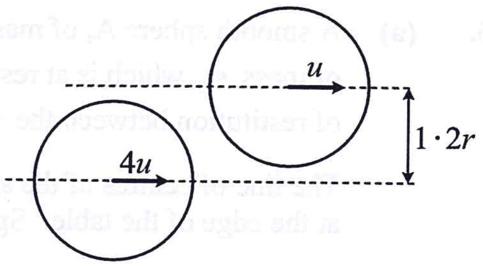
<u>Collision 2:</u> Spheres and cushion

$$v_3=-e_2v_2$$
$$v_3=\frac{-e_2m_1(1+e_1)}{m_1+m_2}u \quad \text{... (5m)}$$

There will be no further collision if

$$v_1<v_3$$
$$\frac{m_1-e_1m_2}{m_1+m_2}u<\frac{-e_2m_1(1+e_1)}{m_1+m_2}u$$
$$m_1-e_1m_2<-e_2\left(m_1+e_1m_1\right)$$
$$m_1+e_2m_1+e_1e_2m_1<e_1m_2$$
$$m_1(1+e_2+e_1e_2)<e_1m_2 \quad \text{... (5m)}$$

(b) Two smooth spheres, each of mass m and radius r, collide while travelling on a smooth horizontal plane. Before impact, the speeds of the spheres are u and $4u$ respectively, and the spheres are moving in the same direction along parallel lines, a distance $1 \cdot 2r$ apart.

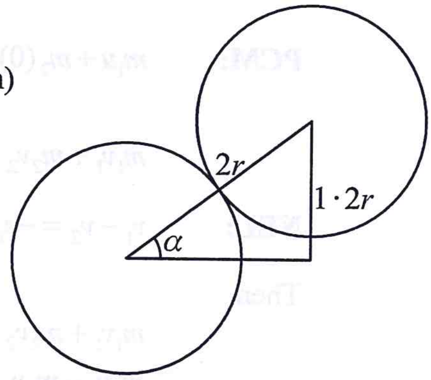


The coefficient of restitution between the spheres is $\dfrac{1}{2}$.

Find the angle between their directions of motion after impact, correct to the nearest degree. **(25)**

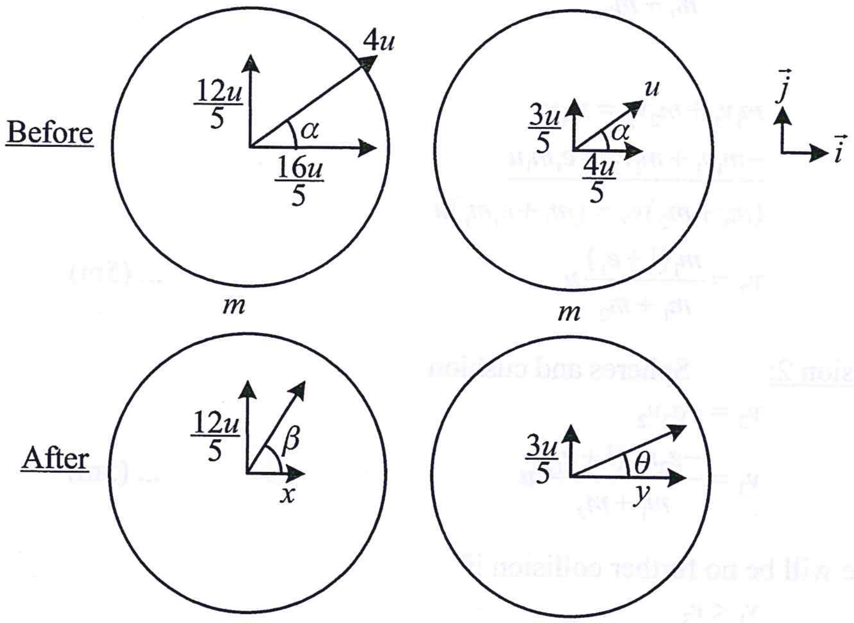
From the diagram

$$\sin\alpha = \frac{1 \cdot 2r}{2r} = \frac{3}{5}$$ ... (5m)



Thus $\cos\alpha = \dfrac{4}{5}$ and $\tan\alpha = \dfrac{3}{4}$.

Let β and θ be the directions of motion of the two spheres after impact.



PCM $(\vec{i})$: $mx + my = m\left(\dfrac{16u}{5}\right) + m\left(\dfrac{4u}{5}\right)$ ... (5m)

$x + y = 4u$ **...1**

NEL $(\vec{i})$: $x-y=-\frac{1}{2}\left(\frac{16u}{5}-\frac{4u}{5}\right)$ ... (5m)

$x-y=-\frac{6u}{5}$ **...2**

Adding **1** and **2**,

$2x=\frac{14u}{5}$

$x=\frac{7u}{5}$

and

$\frac{7u}{5}+y=4u$

$y=\frac{13u}{5}$ ... (5m)

then

$\tan\beta=\frac{\frac{12u}{5}}{\frac{7u}{5}}=\frac{12}{7}$

$\beta=59\cdot74°$

and

$\tan\theta=\frac{\frac{3u}{5}}{\frac{13u}{5}}=\frac{3}{13}$

$\theta=12\cdot99°$

Then the angle between their directions of motion

$=59\cdot74°-12\cdot99°$

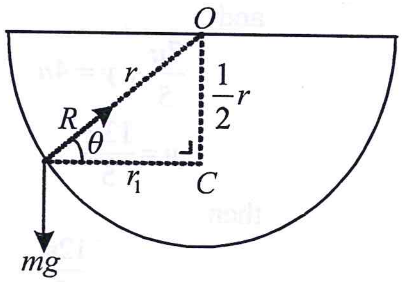
$=47°$ ... (5m)

6. **(a)** A particle P is moving at a constant speed on the inner surface of a smooth sphere of radius r.

The particle is describing horizontal circles $\frac{1}{2}r$ below the centre of the sphere.

Prove that the speed of the particle is $\frac{1}{2}\sqrt{6gr}$. **(25)**

From the diagram,



$$\sin\theta = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$$

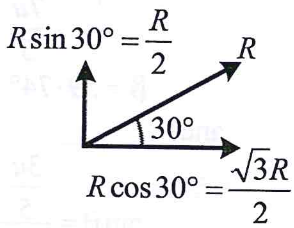
$$\theta = 30° \qquad \text{... (5m)}$$

Then

$$r_1 = r\cos 30°$$

$$r_1 = \frac{\sqrt{3}r}{2} \qquad \text{... (5m)}$$

Then



$$\uparrow = \downarrow : \frac{R}{2} = mg \qquad \text{... (5m)}$$

$$R = 2mg$$

Circular motion:

$$\frac{mv^2}{r_1} = \frac{\sqrt{3}R}{2} \qquad \text{... (5m)}$$

$$mv^2 = \frac{\sqrt{3}}{2}(2mg)\left(\frac{\sqrt{3}r}{2}\right)$$

$$v^2 = \frac{3gr}{2}$$

$$v^2 = \frac{6gr}{4}$$

$$v = \frac{1}{2}\sqrt{6gr} \qquad \text{... (5m)}$$

(b) A particle moves with simple harmonic motion in a straight line. It has velocities of 4 m s^{-1} and 2 m s^{-1} when it is at distances of 1 m and 2 m respectively from the centre of the motion.

(i) Find the amplitude and the periodic time of the motion. **(15)**

$v = 4$ when $x = 1$:

$v = \omega\sqrt{a^2 - x^2}$

$4 = \omega\sqrt{a^2 - 1}$ **...1**

$v = 2$ when $x = 2$:

$2 = \omega\sqrt{a^2 - 4}$ **...2** ... (5m)

Dividing **1** by **2**:

$$\frac{4}{2} = \frac{\omega\sqrt{a^2-1}}{\omega\sqrt{a^2-4}}$$

$$2 = \frac{\sqrt{a^2-1}}{\sqrt{a^2-4}}$$

$$4 = \frac{a^2-1}{a^2-4}$$

$$4a^2 - 16 = a^2 - 1$$

$$3a^2 = 15$$

$$a^2 = 5$$

$a = \sqrt{5}$ m ... (5m)

From **2**,

$$2 = \omega\sqrt{5-4}$$

$$\omega = 2$$

and

$T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ s ... (5m)

(ii) Calculate the least time taken for the particle to travel from a position of rest to a position where its velocity is 2 m s^{-1}. **(10)**

$v = 0$ when $x = a = \sqrt{5}$

Let $x = a\cos\omega t$

$x = \sqrt{5}\cos 2t$

$v = 2$ when $x = 2$

$2 = \sqrt{5}\cos 2t$ ... (5m)

$$\cos 2t = \frac{2}{\sqrt{5}} = 0{\cdot}8944$$

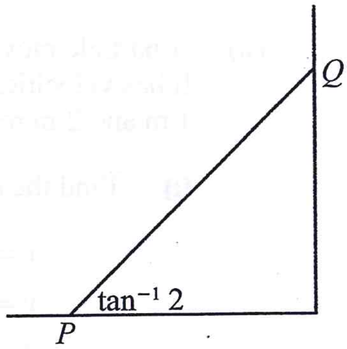
$$2t = 0{\cdot}4636$$

$t = 0{\cdot}23$ s ... (5m)

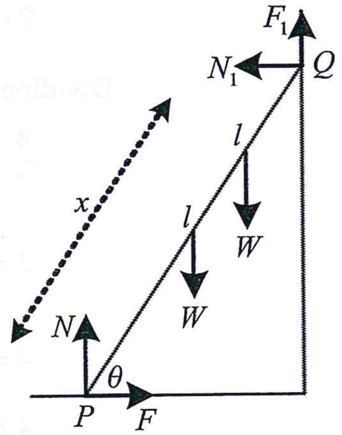
7. **(a)** A uniform ladder, of weight W and length $2l$, rests with its lower end, P, on rough horizontal ground. Its upper end, Q, is in contact with a rough vertical wall.

At both P and Q, the coefficient of friction is $\frac{1}{3}$.

The ladder makes an angle of $\tan^{-1}2$ to the horizontal.



Express in terms of l, the distance that a person of weight W can safely climb before the ladder begins to slip. **(25)**



$\theta = \tan^{-1}2$

$\tan\theta = 2$

Then

$\uparrow = \downarrow$: $\quad N + F_1 = 2W$

$\leftarrow = \rightarrow$: $\quad F = N_1$ ... (5m)

Limiting friction: $\quad F = \frac{1}{3}N$

$F_1 = \frac{1}{3}N_1$

Let x be the distance that the person can ascend.

Taking moments about P:

$$N_1.2l\sin\theta + F_1.2l\cos\theta = W.l\cos\theta + W.x\cos\theta \quad \text{... (5m)}$$

$$2lN_1\tan\theta + 2lF_1 = Wl + Wx$$

$$4lN_1 + 2lF_1 = Wl + Wx$$

then $\quad F_1 = \frac{1}{3}N_1 = \frac{1}{3}F = \frac{1}{9}N$

thus $\quad N + \frac{1}{9}N = 2W$

$$9N + N = 18W$$

$$10N = 18W$$

$$N = \frac{9W}{5} \quad \text{... (5m)}$$

and $\quad F_1 = \frac{W}{5},\ N_1 = \frac{3W}{5}$ ... (5m)

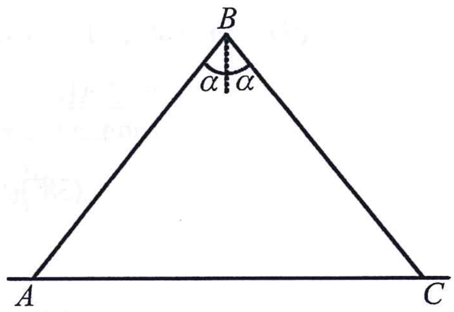
then $\quad 4l\left(\frac{3W}{5}\right) + 2l\left(\frac{W}{5}\right) = Wl + Wx$

$$12l + 2l = 5l + 5x$$

$$9l = 5x$$

$$x = \frac{9l}{5} \quad \text{... (5m)}$$

(b) Two uniform rods, AB and BC, each of length $2l$ are smoothly jointed at B. The weight of AB is $3W$ and the weight of BC is $5W$. The rods stand in equilibrium with the ends A and C on rough horizontal ground, with each rod making an angle α with the vertical.



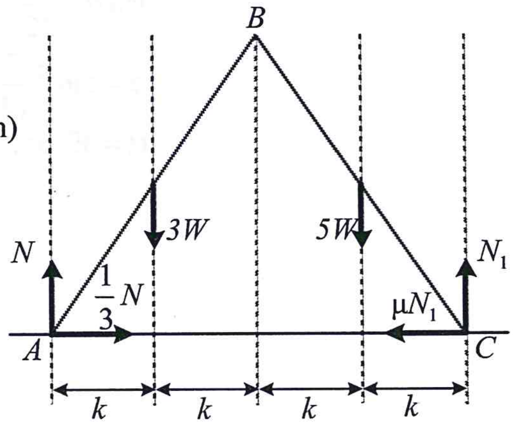
The coefficient of friction between A and the ground is $\frac{1}{3}$, while the coefficient of friction between C and the ground is μ. The angle α is increased until both rods are on the point of slipping.

Find **(i)** the value of μ. **(15)**

<u>Structure ABC</u>

$\uparrow = \downarrow$: $N + N_1 = 8W$

$\leftarrow = \rightarrow$: $\frac{1}{3}N = \mu N_1$ ... (5m)



Moments about A:

$$(3W)(k) + (5W)(3k) = (N_1)(4k)$$

$$18W = 4N_1$$

$$N_1 = \frac{9}{2}W$$

Then

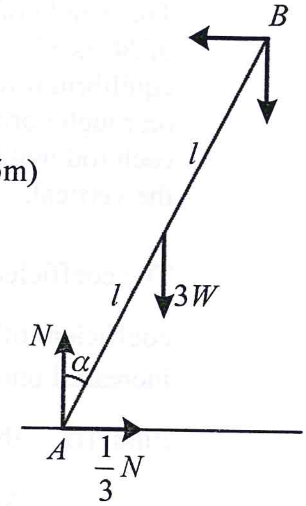
$$N = \frac{7}{2}W \quad \text{... (5m)}$$

and

$$\frac{1}{3}\left(\frac{7}{2}W\right) = \mu\left(\frac{9}{2}W\right)$$

$$\mu = \frac{7}{27} \quad \text{... (5m)}$$

(ii) the value of α when the rods are about to slip. **(10)**



Rod AB

Moments about B

$$(3W)(l\sin\alpha)+\left(\frac{1}{3}N\right)(2l\cos\alpha)$$

$$=(N)(2l\sin\alpha) \qquad \text{... (5m)}$$

$$(3W)(\tan\alpha)+\left(\frac{7W}{6}\right)(2)=\left(\frac{7W}{2}\right)(2\tan\alpha)$$

$$3\tan\alpha+\frac{7}{3}=7\tan\alpha$$

$$\frac{7}{3}=4\tan\alpha$$

$$\tan\alpha=\frac{7}{12}$$

$$\alpha=\tan^{-1}\frac{7}{12} \qquad \text{... (5m)}$$

$$\alpha=30\cdot3^\circ.$$

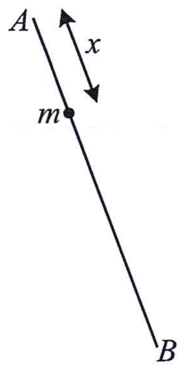
8. **(a)** Prove that the moment of inertia of a uniform rod of mass m and length $2l$ about an axis through its centre perpendicular to the rod is $\frac{1}{3}ml^2$. **(20)**

Standard Proof

Moment of mass element	... (5m)
Moment of body	... (5m)
Integral	... (5m)
Deduce	... (5m)

(b) A uniform rod AB of mass m and length $2l$ has a particle of mass m attached at a distance $x > 0$ from A. The system is free to rotate about a horizontal axis through A perpendicular to the rod.

When the system makes small oscillations about the horizontal axis through A, the length of the equivalent simple pendulum is $\frac{4l}{3}$.

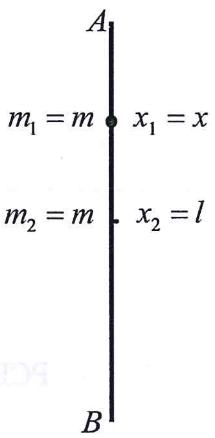


(i) Express x in terms of l. **(15)**

Let h be the distance from A to the centre of mass of the system.

$$h = \frac{(m)(x) + (m)(l)}{m + m}$$

$$h = \frac{x + l}{2}$$



Let I be the moment of inertia of the system about the horizontal axis through A.

Then

$$I = I_{\text{rod}} + I_{\text{particle}}$$

$$I = \frac{4}{3}ml^2 + mx^2 \qquad \text{... (5m)}$$

Given

$$\frac{I}{Mh} = \frac{4l}{3}$$

$$\frac{\frac{4}{3}ml^2 + mx^2}{(2m)\left(\frac{x+l}{2}\right)} = \frac{4l}{3} \qquad \text{... (5m)}$$

Further answers overleaf

$$\frac{4}{3}l^2 + x^2 = \frac{4l}{3}(x+l)$$
$$4l^2 + 3x^2 = 4lx + 4l^2$$
$$3x^2 = 4lx$$
$$3x = 4l$$
$$x = \frac{4l}{3}$$ ... (5m)

(ii) If the system is released from rest with *AB* horizontal, find the speed of *B* when it is vertically below *A*. **(15)**

When *AB* is horizontal,

$$KE_1 = 0$$
$$PE_1 = (2m)g\left(\frac{\frac{4l}{3}+l}{2}\right)$$
$$= mg\left(\frac{7l}{3}\right)$$
$$= \frac{7mgl}{3}$$

When *B* is vertically below *A*,

$$KE_1 = \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}\left(\frac{4}{3}ml^2 + m\left(\frac{4l}{3}\right)^2\right)\omega^2$$
$$= \frac{14}{9}ml^2\omega^2$$
$$PE_2 = 0$$

PCE: $KE_1 + PE_1 = KE_2 + PE_2$

$$\frac{7mgl}{3} = \frac{14}{9}ml^2\omega^2$$ ... (5m)
$$l\omega^2 = \frac{3g}{2}$$
$$\omega^2 = \frac{3g}{2l}$$
$$\omega = \sqrt{\frac{3g}{2l}}$$ ... (5m)

Let v be the velocity of B at its lowest point. Then

$$v = r\omega$$

$$v = (2l)\sqrt{\frac{3g}{2l}}$$

$$v = \sqrt{4l^2 \times \frac{3g}{2l}}$$

$$v = \sqrt{6gl}$$ ... (5m)

9. **(a)** 275 cm^3 of a liquid of relative density $2{\cdot}1$ is mixed with $V \text{ cm}^3$ of another liquid of relative density $3{\cdot}5$.

If there is no contraction of volume, and the relative density of the mixture is $2{\cdot}95$ find the value of V. **(15)**

	1	2	1 and 2
m			
V	275×10^{-6}	$V\times10^{-6}$	
ρ			
s	$2{\cdot}1$	$3{\cdot}5$	$2{\cdot}95$

Then

$$s=\frac{s_1V_1+s_2V_2}{V_1+V_2}$$

$$2{\cdot}95=\frac{(2{\cdot}1)(275\times10^{-6})+(3{\cdot}5)(V\times10^{-6})}{(275\times10^{-6})+(V\times10^{-6})}$$ ... (5m)

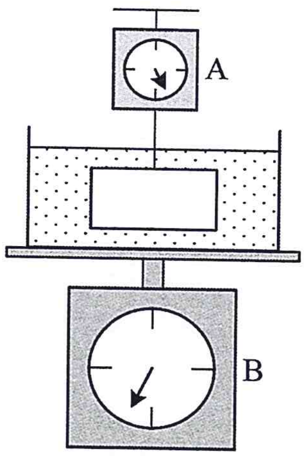
$$2{\cdot}95=\frac{577{\cdot}5+3{\cdot}5V}{275+V}$$ ... (5m)

$$811{\cdot}25+2{\cdot}95V=577{\cdot}5+3{\cdot}5V$$

$$233{\cdot}75=0{\cdot}55V$$

$$V=425$$ ... (5m)

(b) A block of mass $2{\cdot}1$ kg, and relative density $4{\cdot}2$, is held suspended by a string attached to a scale A. The block is completely immersed in 1500 cm^3 of a liquid of relative density $1{\cdot}2$ contained in a cylindrical beaker of mass $0{\cdot}7$ kg. The beaker sits on another scale B.



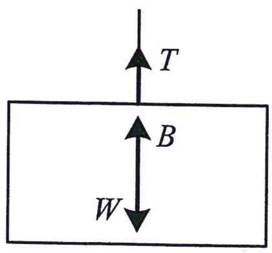
(i) If scale A registers x kg, find the value of x. **(15)**

Let W be the weight and B be the buoyancy.

$$W=2{\cdot}1g$$

and

$$B=\frac{s_LW}{s}$$



$$= \frac{1\cdot2(2\cdot1g)}{4\cdot2} \quad \text{... (5m)}$$
$$= 0\cdot6g$$

In equilibrium:

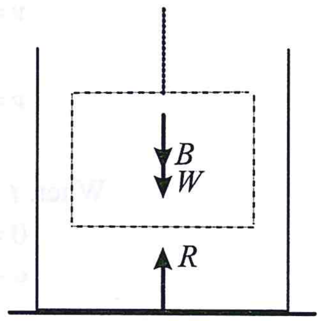
$$T + B = W$$
$$T + 0\cdot6g = 2\cdot1g \quad \text{... (5m)}$$
$$T = 1\cdot5g$$

If scale A registers a mass of x kg, then $x = 1\cdot5$. ... (5m)

(ii) If scale B registers y kg, find the value of y. **(10)**

Weight of liquid:

$$= 1000(1\cdot2)\left(1500\times10^{-6}\right)g$$
$$= 1\cdot8g$$



Let W be the total weight of the beaker and the liquid. Then

$$W = 0\cdot7g + 1\cdot8g \quad \text{... (5m)}$$
$$= 2\cdot5g$$

In equilibrium:

$$R = W + B$$
$$= 2\cdot5g + 0\cdot6g$$
$$= 3\cdot1g$$

If scale B registers a mass of y kg, then $y = 3\cdot1$. ... (5m)

(iii) The radius of the beaker is 10 cm. Find the height, in cm, of the liquid in the beaker, correct to two decimal places. **(10)**

Let V be the volume of the block. Then

$$m = 1000sV$$
$$2\cdot1 = 1000(4\cdot2)V$$
$$V = 0\cdot0005 \text{ m}^3$$
$$= 500 \text{ cm}^3 \quad \text{... (5m)}$$

Total volume of liquid and block

$$= 1500 + 500$$
$$= 2000 \text{ cm}^3$$

Let h cm be the height of the liquid. Then

$$\pi(10)^2 h = 2000$$
$$h = 6\cdot37 \text{ cm} \quad \text{... (5m)}$$

10. (a) A particle moving in a straight line experiences an acceleration of $-4\cos\frac{1}{6}t$ cm s^{-2} at time t seconds. At time $t=0$ the particle is at rest and has a displacement of 144 cm relative to a fixed point O on the line.

(i) Find the first positive time that the particle reaches the point O. **(20)**

$$\frac{d^2x}{dt^2}=-4\cos\frac{1}{6}t$$

$$\frac{dv}{dt}=-4\cos\frac{1}{6}t$$

$$\int dv=-4\int\cos\frac{1}{6}t\,dt$$

$$v=-4\left[6\sin\frac{1}{6}t\right]+c \qquad \text{... (5m)}$$

$$v=-24\sin\frac{1}{6}t+c$$

When $t=0,\ v=0$:

$$0=-24\sin 0+c$$
$$c=0$$

The unique solution is

$$v=-24\sin\frac{1}{6}t \qquad \text{... (5m)}$$

Then

$$\frac{dx}{dt}=-24\sin\frac{1}{6}t$$

$$\int dx=-24\int\sin\frac{1}{6}t\,dt$$

$$x=-24\left[-6\cos\frac{1}{6}t\right]+d$$

$$x=144\cos\frac{1}{6}t+d$$

When $t=0,\ x=144$

$$144=144\cos 0+d$$
$$144=144+d$$
$$d=0$$

The unique solution is:

$$x=144\cos\frac{1}{6}t \qquad \text{... (5m)}$$

When $x=0$:

$$0=144\cos\frac{1}{6}t$$

$$\cos\frac{1}{6}t = 0$$
$$\frac{1}{6}t = \frac{\pi}{2}$$
$$t = 3\pi \text{ s}$$
$$t = 9.425$$ ... (5m)

(ii) Show that the particle is moving with simple harmonic motion. **(5)**

$$\text{acc} = -4\cos\frac{1}{6}t$$
$$\text{acc} = -4\left[\frac{x}{144}\right]$$
$$\text{acc} = -\frac{1}{36}x$$

As this is in the form

$$\text{acc} = -\omega^2 x$$ ... (5m)

the particle is moving with simple harmonic motion.

(b) A particle moving in a straight line of mass m is acted upon by a force of magnitude $\frac{2m}{x^5}$ directed away from a fixed point O on the line, where x is the distance of the particle from O.

The particle starts from rest at a distance d from O.
Show that the velocity of the particle tends to a limit of $\frac{1}{d^2}$. **(25)**

$$m\frac{d^2x}{dt^2} = \frac{2m}{x^5}$$
$$v\frac{dv}{dx} = \frac{2}{x^5}$$ ... (5m)
$$\int v\,dv = 2\int x^{-5}\,dx$$
$$\frac{1}{2}v^2 = 2\left[\frac{1}{-4}x^{-4}\right] + \frac{c}{2}$$
$$\frac{1}{2}v^2 = -\frac{1}{2}.\frac{1}{x^4} + \frac{c}{2}$$

Further answers overleaf

$$v^2 = -\frac{1}{x^4} + c$$ ... (5m)

$v = 0$ when $x = d$:

$$0 = -\frac{1}{d^4} + c$$

$$c = \frac{1}{d^4}$$

Unique solution:

$$v^2 = -\frac{1}{x^4} + \frac{1}{d^4}$$

$$v = \sqrt{\frac{1}{d^4} - \frac{1}{x^4}}$$ ... (5m)

In the limit as $x \to \infty$,

$$\lim_{x\to\infty} v = \lim_{x\to\infty} \sqrt{\frac{1}{d^4} - \frac{1}{x^4}}$$ ... (5m)

$$= \sqrt{\frac{1}{d^4} - \lim_{x\to\infty} \frac{1}{x^4}}$$

$$= \sqrt{\frac{1}{d^4} - 0}$$

$$= \frac{1}{d^2}.$$ ... (5m)